Probability and Random Processes

ECS 315

Asst. Prof. Dr. Prapun Suksompong

prapun@siit.tu.ac.th

11 Multiple Random Variables

Office Hours:

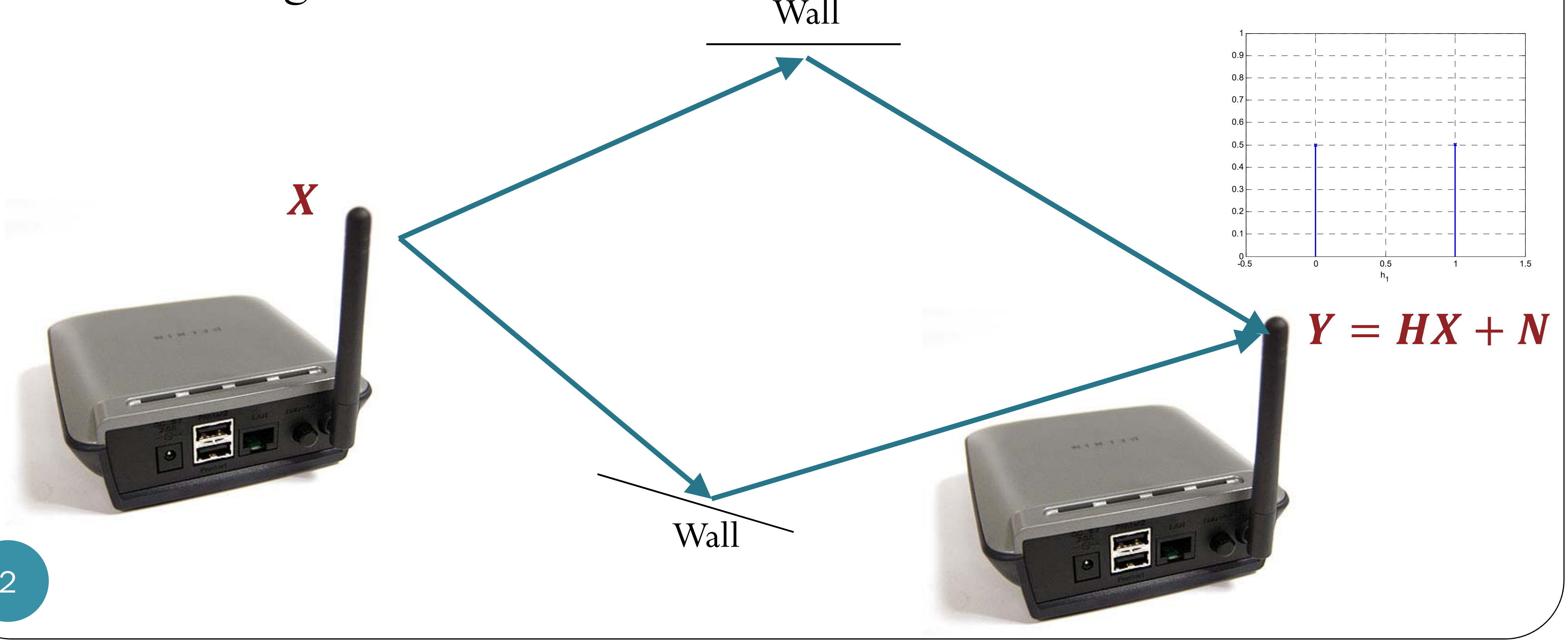
Check Google Calendar on the course website.

Dr.Prapun's Office:

6th floor of Sirindhralai building, BKD

SISO Wireless Communications

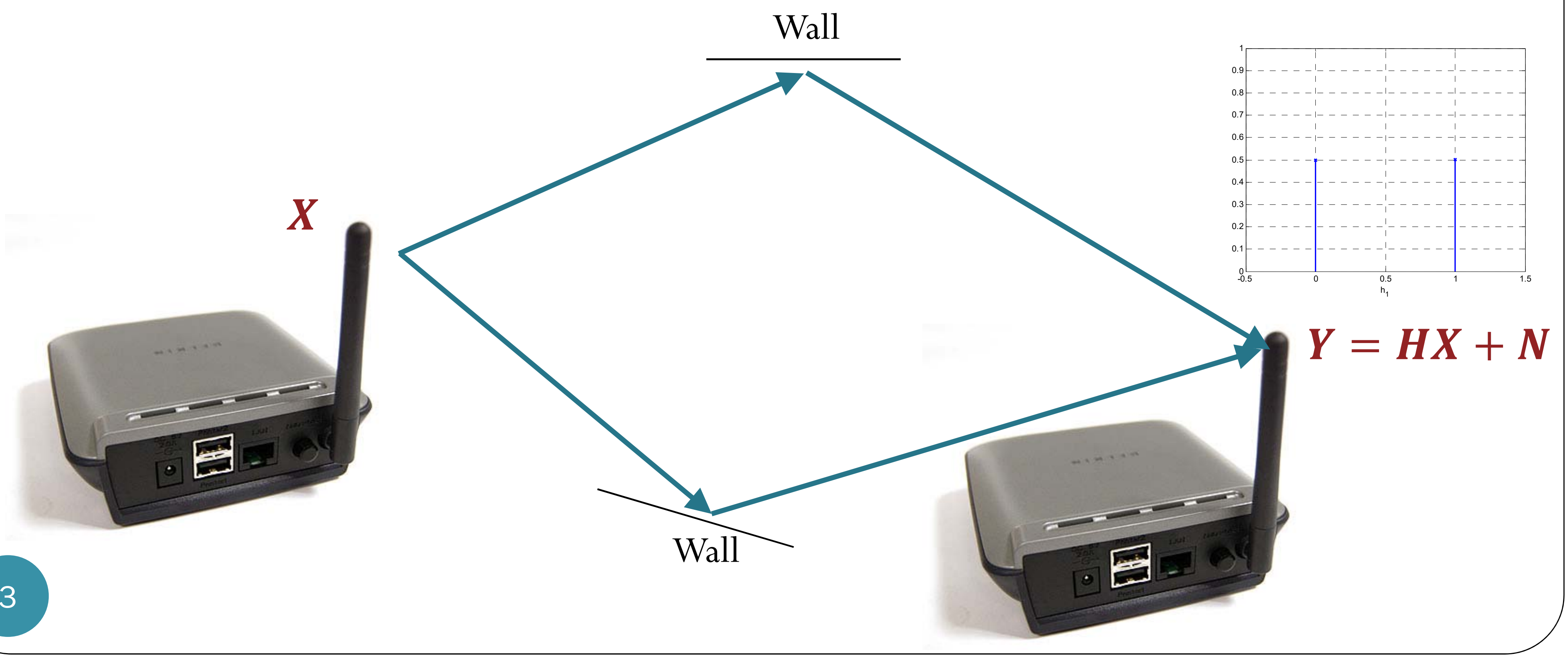
- Multipath propagation
- At the receiver, multiple copies of the signal may be combined constructively or destructively.
- Fading



SISO Wireless Communications

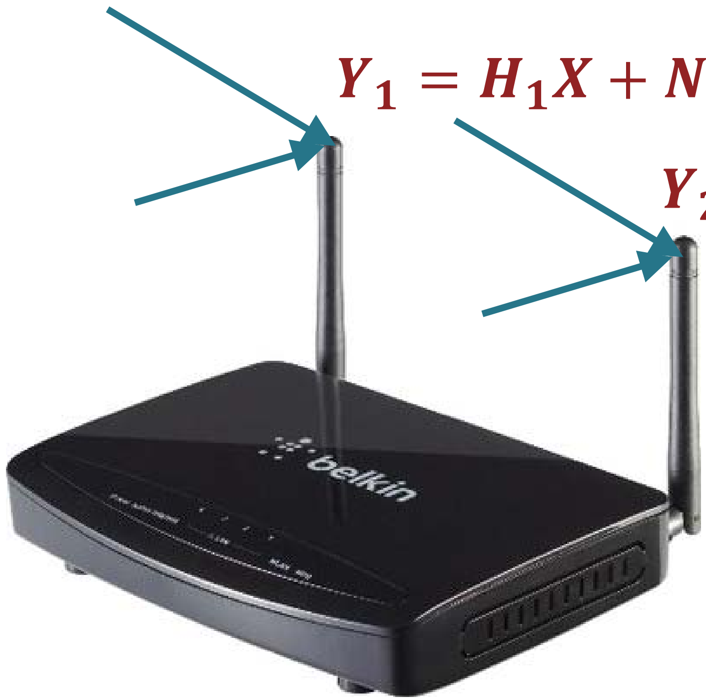
- H = channel coefficient (quality)
- For simplicity, let's assume two possible values for H: good (1) or bad (0).

  Its value is random. H: 0 1 0 0 1 1 1 1 0 1 …



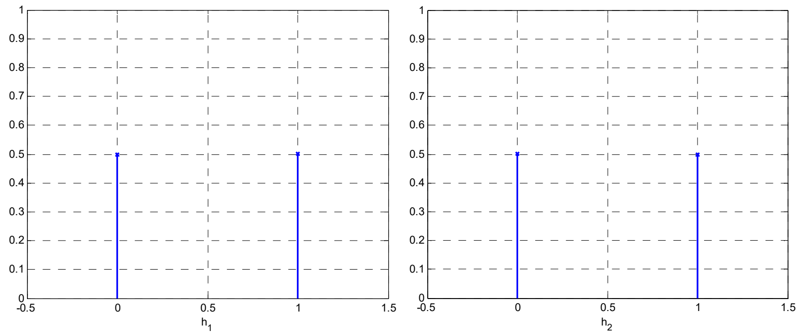
MIMO Wireless Communications

- Here, there are **two** antennas to receive the signals
- Use the antenna that receive stronger signal (less fading; better channel condition)



$$Y_1 = H_1 X + N_1$$

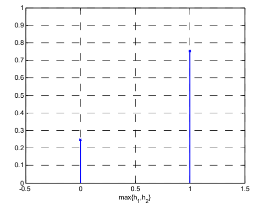
$$Y_2 = H_2 X + N_2$$



H_1: 0 1 0 0 1 1 1 1 0 1 …

H_2: 0 1 1 0 0 0 0 1 0 1 …

H_{used}: 0 1 1 0 1 1 1 1 0 1 …



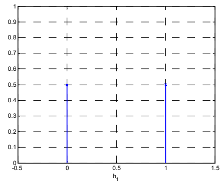
MIMO Wireless Communications

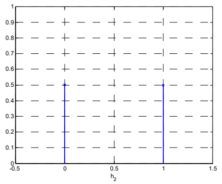
- Here, there are **three** antennas to receive the signals
- Use the antenna that receive the strongest signal (least fading; best channel condition)

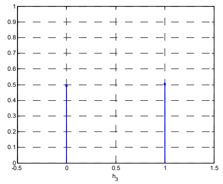
$$Y_1 = H_1 X + N_1$$

$$Y_2 = H_2 X + N_2$$

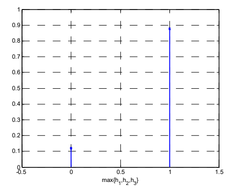
$$Y_3 = H_3 X + N_3$$







H_1: 0	1	0	0	1	1	1	1	0	1	…
H_2: 0	1	1	0	0	0	0	1	0	1	…
H_3: 0	1	1	1	1	0	1	0	1	1	…
H_{used}: 0	1	1	1	1	1	1	1	1	1	…

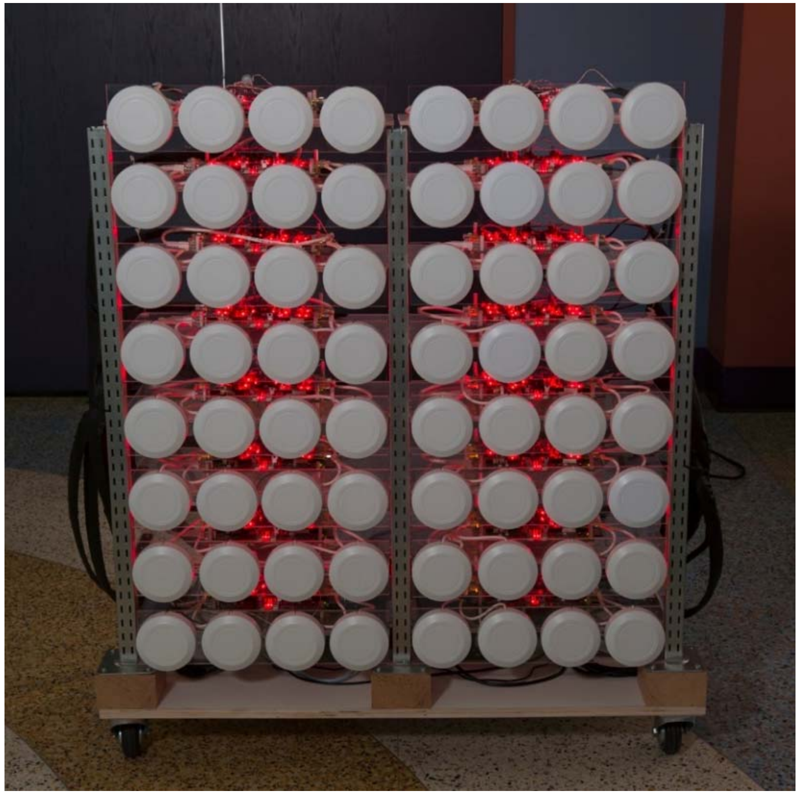


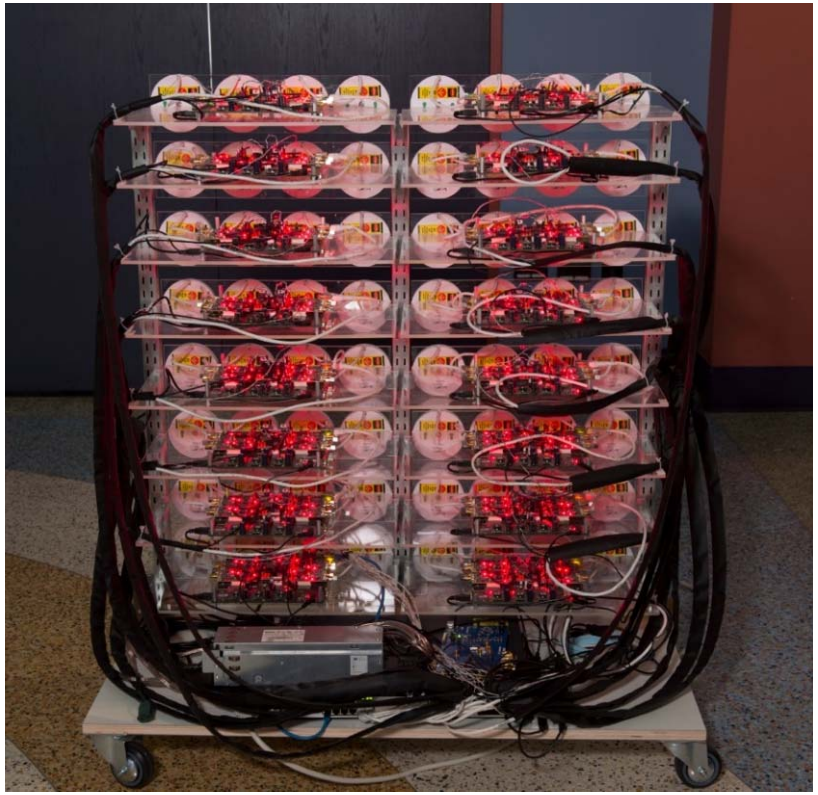
MIMO Communications

- Of course, even more antennas is also possible.

Very Large MIMO Systems

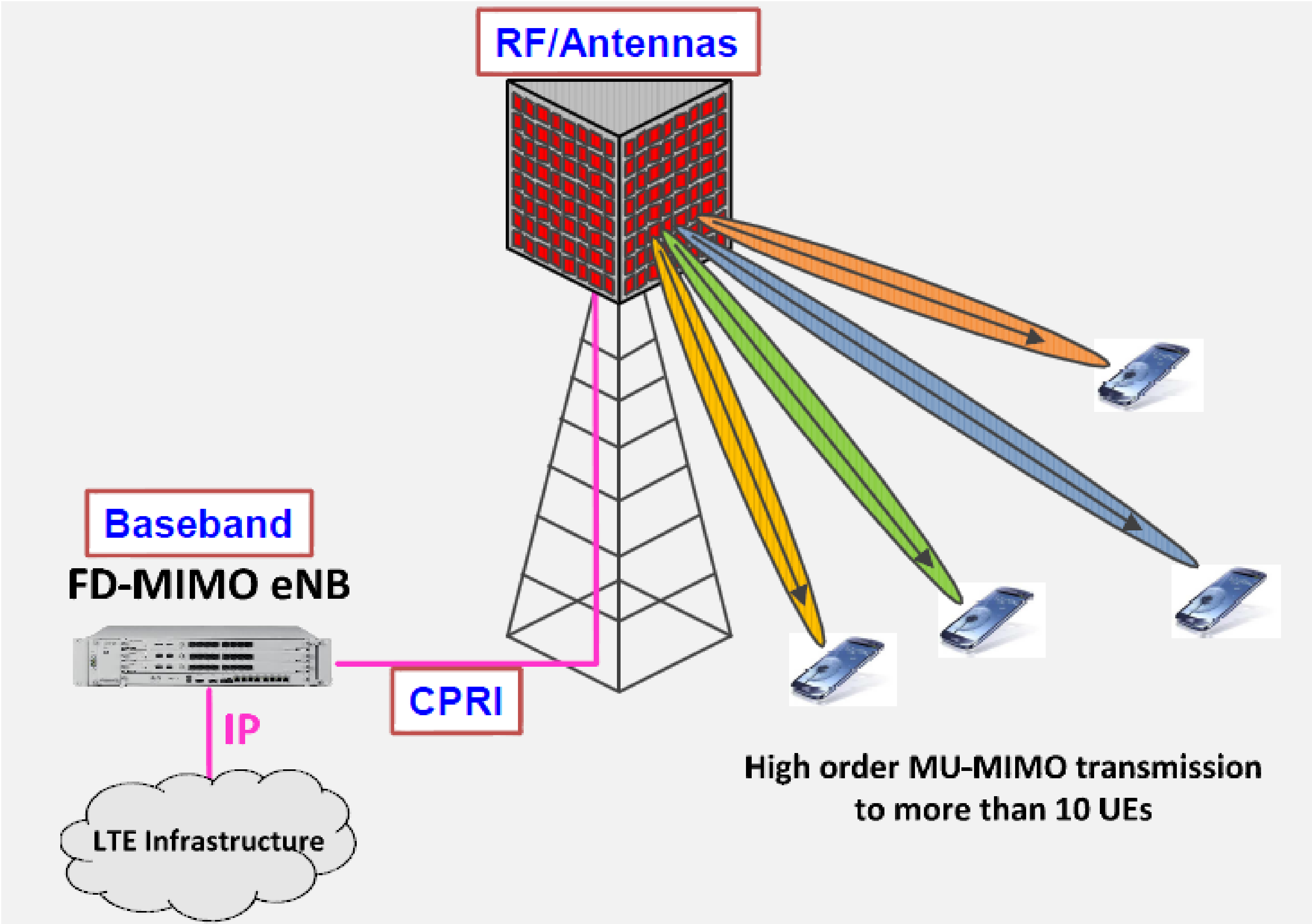
- “antenna array”





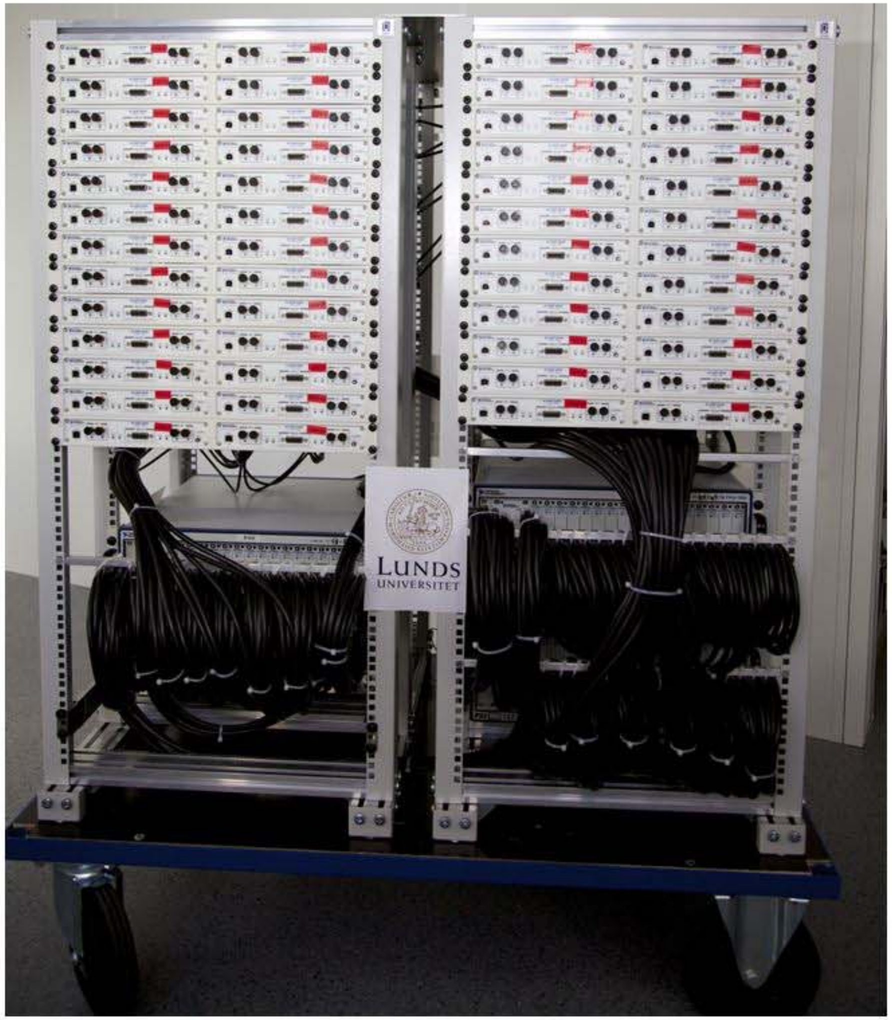
[http://argos.rice.edu/]

Very Large MIMO Systems



[http://www.slideshare.net/zahidtg/full-dimension-mimo-and-beamforming-technologies-for-b4g]

Very Large MIMO Systems



[http://www.ni.com/white-paper/52382/en/]

Very Large MIMO Systems

[http://forums.xilinx.com/t5/Xcell-Daily-Blog/Software-defined-radio-dev-platform-for-5G-research-handles-MIMO/ba-p/441754]

Chapter 6 vs. Chapter 11

Joint probability

$$P(\underbrace{A \cap B}_{\text{Joint event}})$$

Joint pmf

$$p_{X,Y}(x,y) = P[X = x, Y = y]$$

$$A = [X = x]$$

$$B = [Y = y]$$

Conditional probability

$$P(A|B) = \frac{P(A \cap B)}{P(B)}$$

$$= \frac{P(B|A)P(A)}{P(B)}$$

Conditional pmf

$$p_{X|Y}(x \mid y) = \frac{p_{X,Y}(x,y)}{p_Y(y)}$$

$$= \frac{p_{Y|X}(y \mid x)p_X(x)}{p_Y(y)}$$

Events A and B are **independent**:

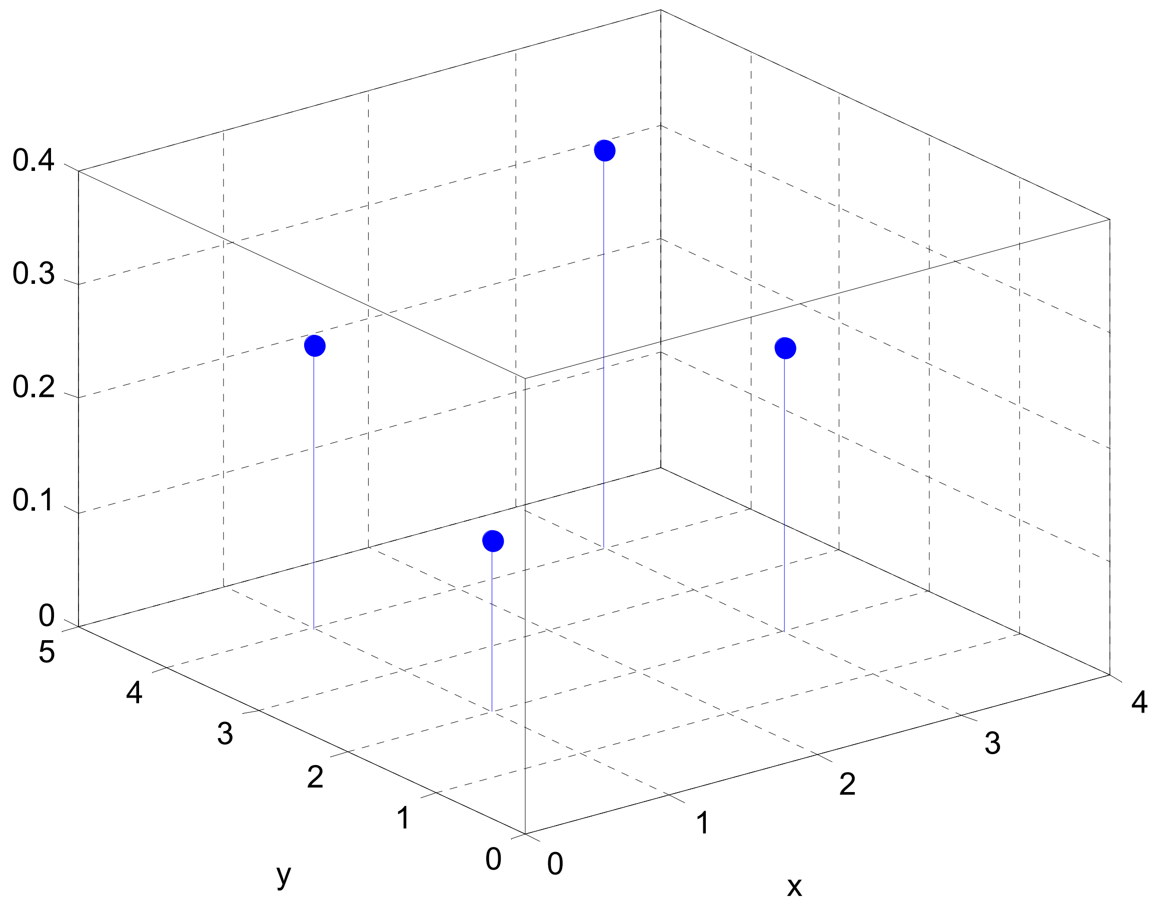
$$P(A \cap B) = P(A)P(B)$$

RVs X and Y are **independent**:

$$p_{X,Y}(x,y) = p_X(x)p_Y(y) \text{ for any } x \text{ and } y$$

Example: small joint pmf matrix Ex. 11.7

```
close all; clear all;
x = [1 3];
y = [2 4];
PXY = [3/20 5/20; 5/20 7/20];

[X Y] = meshgrid(x,y);
X = X.'; Y = Y.';

stem3(X,Y,PXY,'filled')
xlim([0,4])
ylim([0,5])
xlabel('x')
ylabel('y')
```

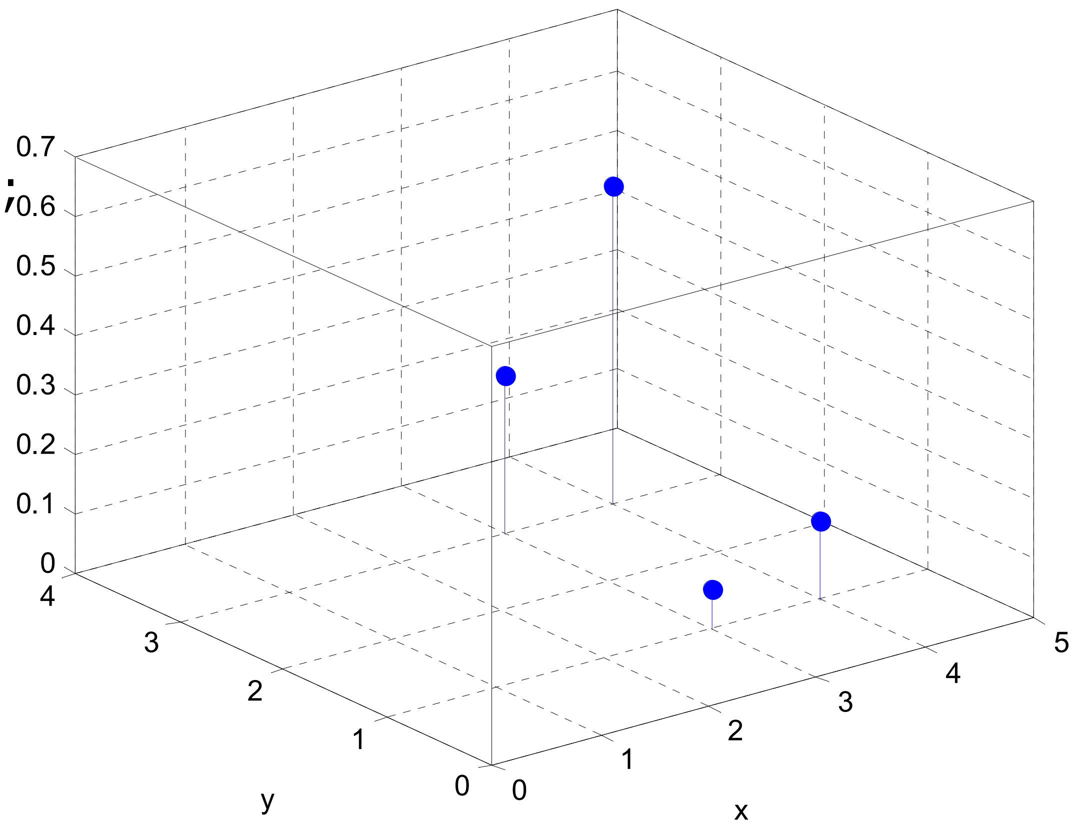


(More)

Example: small joint pmf matrix

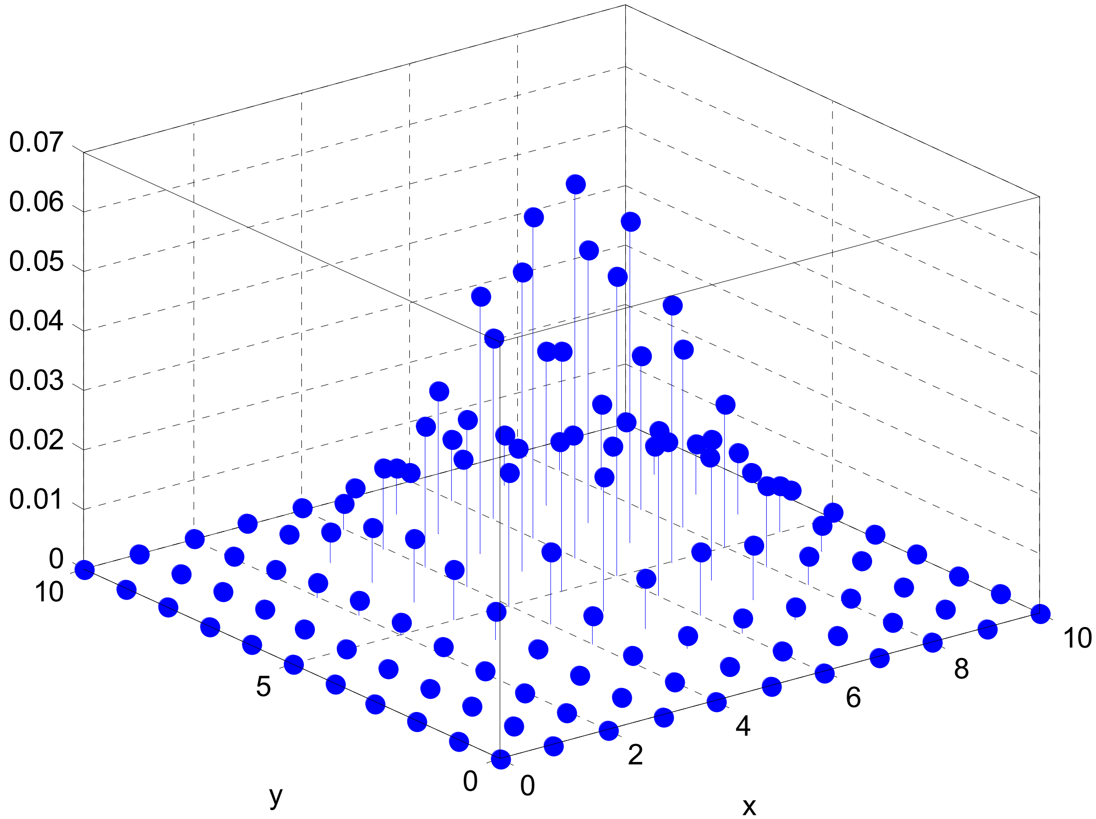
Ex. 11.26

```
close all; clear all;
x = [3 4];
y = [1 3];
PXY = [1/15 4/15; 2/15 8/15];

[X Y] = meshgrid(x,y);
X = X.'; Y = Y.';

stem3(X,Y,PXY,'filled')
xlim([0,5])
ylim([0,4])
xlabel('x')
ylabel('y')
```



Example: large joint pmf matrix

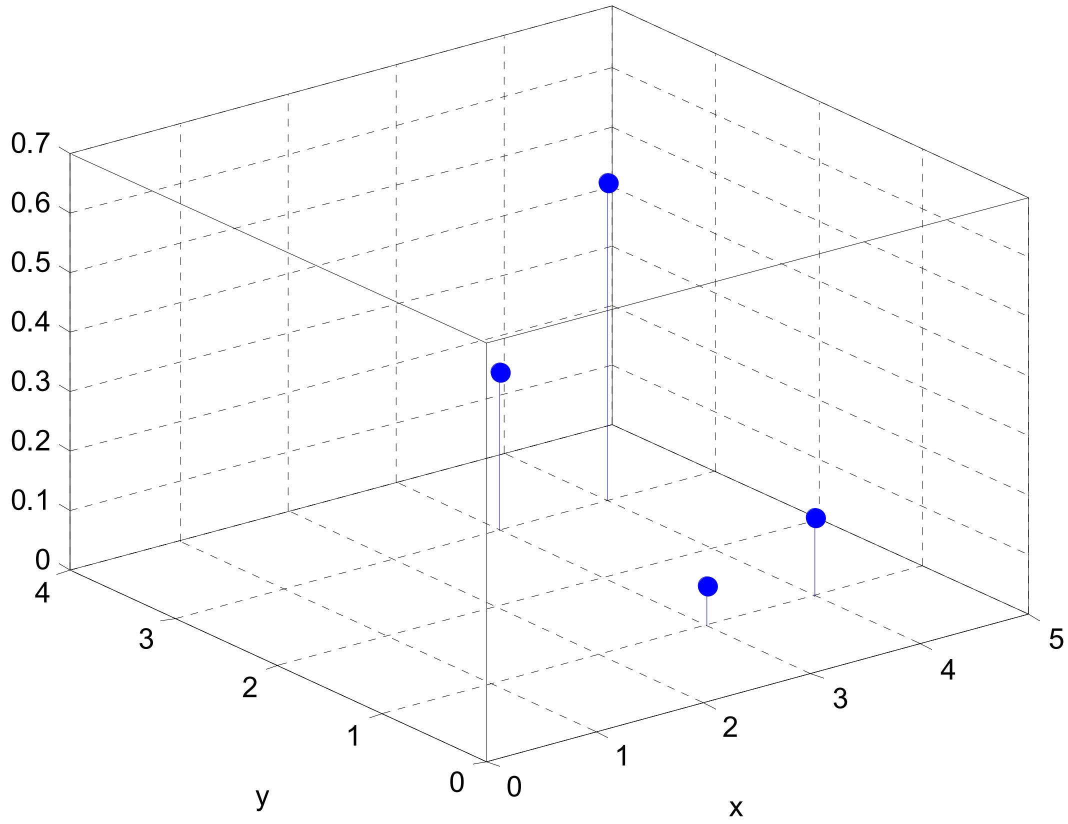
```
close all; clear all;
n = 10; p = 3/5;
x = 0:n;
y = 0:n;

pX = binopdf(x,n,p);
pY = binopdf(y,n,p);

PXY = pX.'*pY;

[X Y] = meshgrid(x,y);
X = X.'; Y = Y.';

stem3(X,Y,PXY, 'filled')
%mesh(X,Y,PXY)
%surf(X,Y,PXY)

xlabel('x')
ylabel('y')
```



Example: small joint pmf matrix Ex. 11.29

$$P_{X,Y} = \begin{array}{c|cc} x \backslash y & 1 & 3 \\ \hline 3 & 1/15 & 4/15 \\ 4 & 2/15 & 8/15 \end{array}$$

```
close all; clear all;
x = [3 4];
y = [1 3];
PXY = [1/15 4/15; 2/15 8/15];

[X Y] = meshgrid(x,y);
X = X.'; Y = Y.';

stem3(X,Y,PXY,'filled')
xlim([0,5])
ylim([0,4])
xlabel('x')
ylabel('y')
```



Evaluation of Probability

- Consider two random variables X and Y.
- Suppose their **joint pmf matrix** is

$P_{X,Y}$

x \ y	2	3	4	5	6
1	0.1	0.1	0	0	0
3	0.1	0	0	0.1	0
4	0	0.1	0.2	0	0
6	0	0	0	0	0.3

$$p_{X,Y}(4,3) = P[X = 4, Y = 3] = 0.1$$

and

Evaluation of Probability

- Consider two random variables X and Y.
- Suppose their **joint pmf matrix** is

$P_{X,Y}$

x \ y	2	3	4	5	6
1	0.1	0.1	0	0	0
3	0.1	0	0	0.1	0
4	0	0.1	0.2	0	0
6	0	0	0	0	0.3

- Find $P[X + Y < 7]$

Evaluation of Probability

- Consider two random variables X and Y.
- Suppose their **joint pmf matrix** is

$P_{X,Y}$

x \ y	2	3	4	5	6
1	0.1	0.1	0	0	0
3	0.1	0	0	0.1	0
4	0	0.1	0.2	0	0
6	0	0	0	0	0.3

- Find $P[X + Y < 7]$

Step 1: Find the pairs (x,y) that satisfy the condition "$x+y < 7$"

One way to do this is to first construct the matrix of $x+y$.

$x + y$

x \ y	2	3	4	5	6
1	3	4	5	6	7
3	5	6	7	8	9
4	6	7	8	9	10
6	8	9	10	11	12

Evaluation of Probability

- Consider two random variables X and Y.
- Suppose their **joint pmf matrix** is

$P_{X,Y}$

x \ y	2	3	4	5	6
1	0.1	0.1	0	0	0
3	0.1	0	0	0.1	0
4	0	0.1	0.2	0	0
6	0	0	0	0	0.3

- Find $P[X+Y<7]$

Step 2: Add the corresponding probabilities from the joint pmf (matrix)

$x+y$

x \ y	2	3	4	5	6
1	3	4	5	6	7
3	5	6	7	8	9
4	6	7	8	9	10
6	8	9	10	11	12

$$P[X+Y<7] = 0.1+0.1+0.1 = 0.3$$

Joint pmf matrix for independent RVs

Command Window

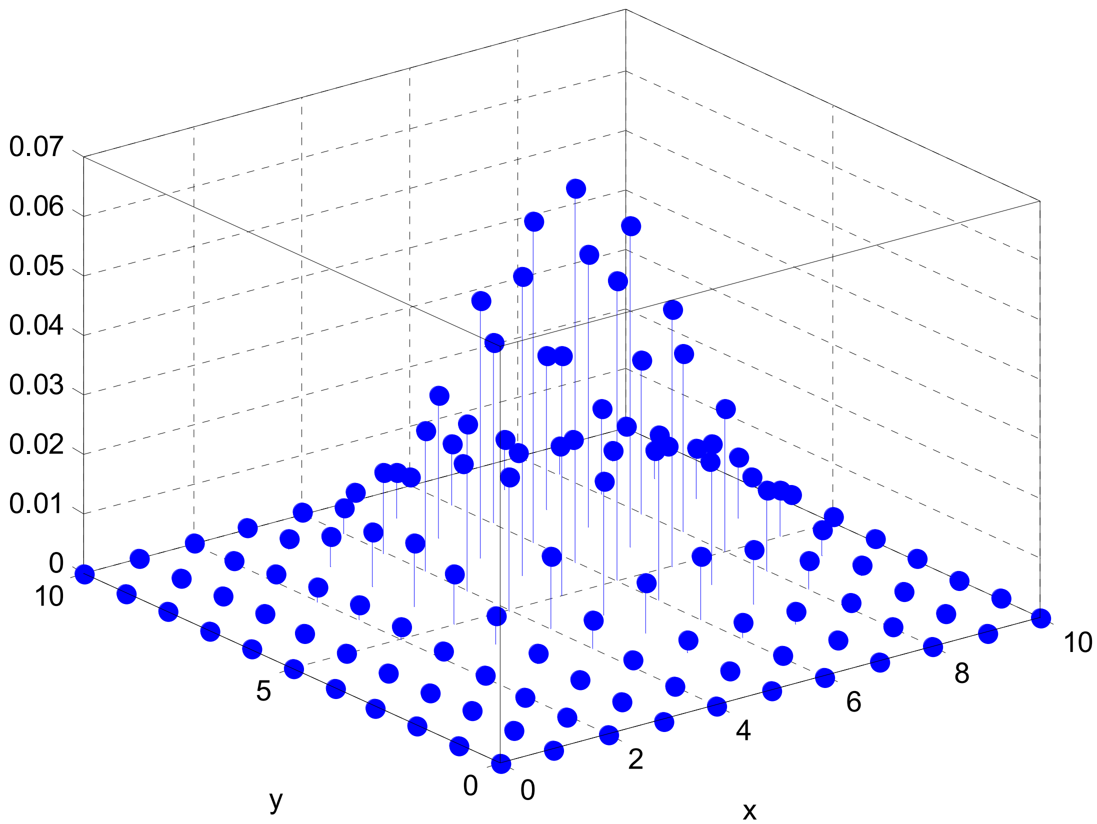
```
>> pX = [1/3 2/3]
pX =
    0.3333    0.6667
>> pY = [1/5 4/5]
pY =
    0.2000    0.8000
>> sym(pX'*pY)
ans =
[ 1/15, 4/15]
[ 2/15, 8/15]
>>
```

Joint pmf for two i.i.d. RVs

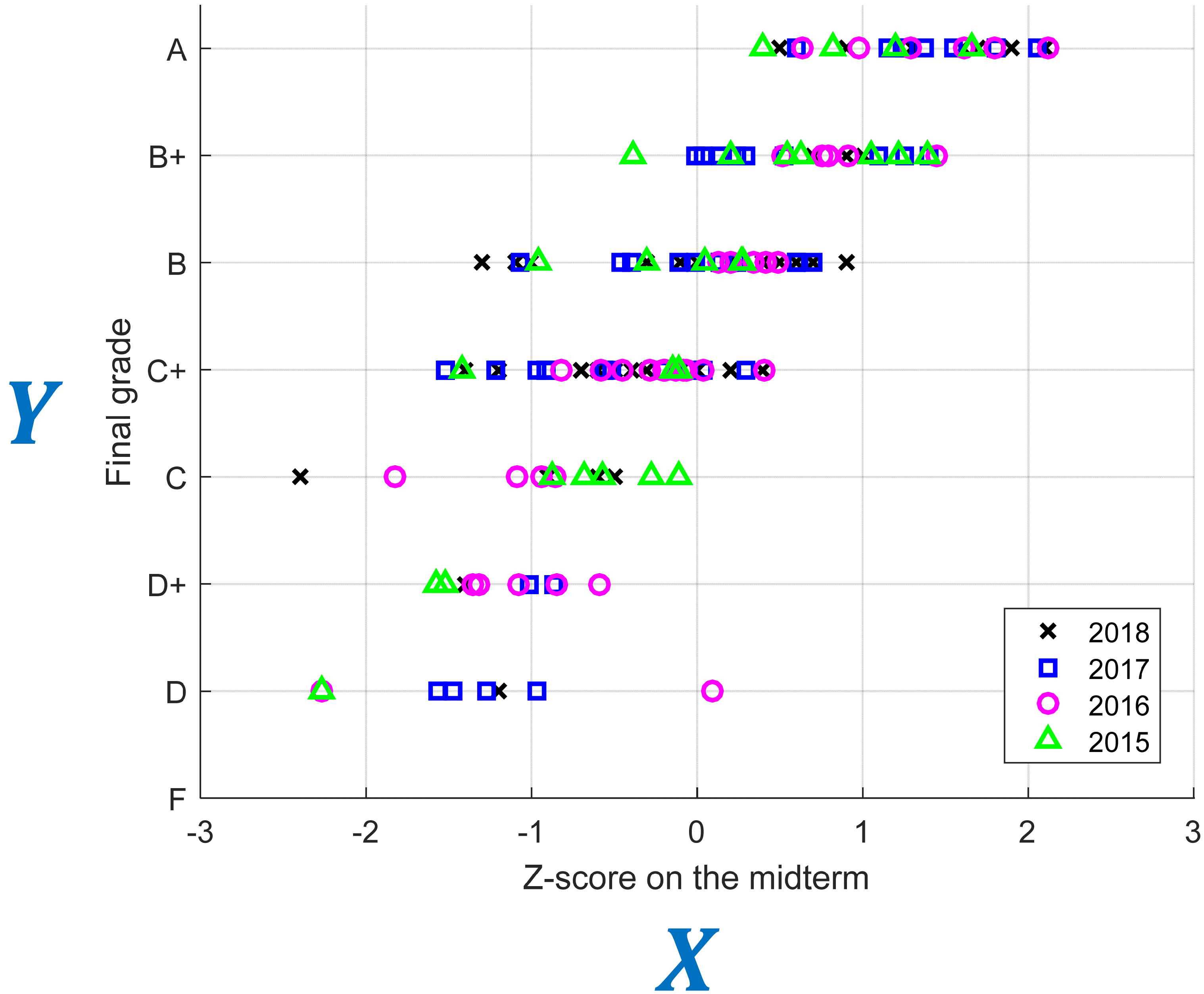
```
close all; clear all;
n = 10; p = 3/5;
x = 0:n;
y = 0:n;

pX = binopdf(x,n,p);
pY = binopdf(y,n,p);

PXY = pX.'*pY;

[X Y] = meshgrid(x,y);
X = X.'; Y = Y.';

stem3(X,Y,PXY, 'filled')
%mesh(X,Y,PXY)
%surf(X,Y,PXY)

xlabel('x')
ylabel('y')
```

Note how the pmfs are multiplied because of the independence.

$$X, Y \overset{\text{i.i.d.}}{\sim} \mathcal{B}\left(10, \frac{3}{5}\right)$$



Dependency

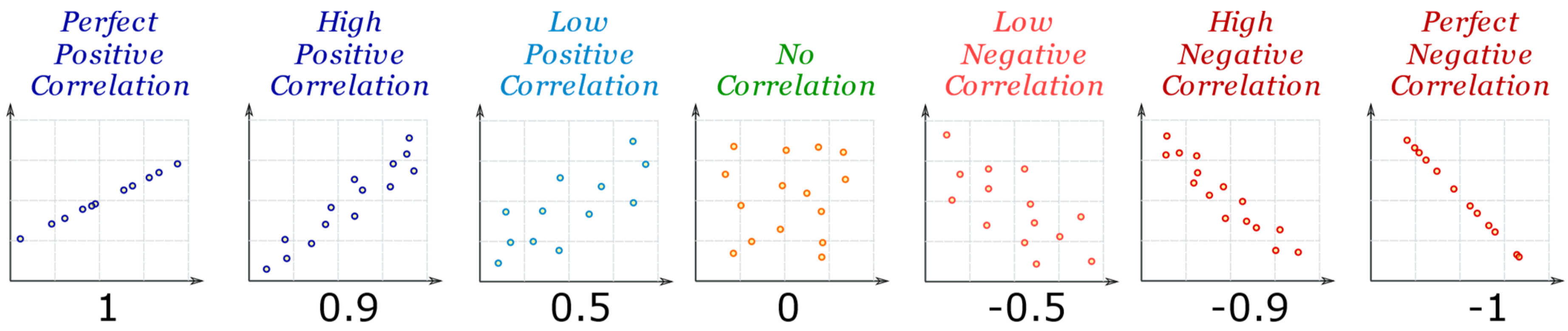


Y
Final grade
A
B+
B
C+
C
D+
D
F
-3
-2
-1
0
1
2
3
Z-score on the midterm
2018
2017
2016
2015
X

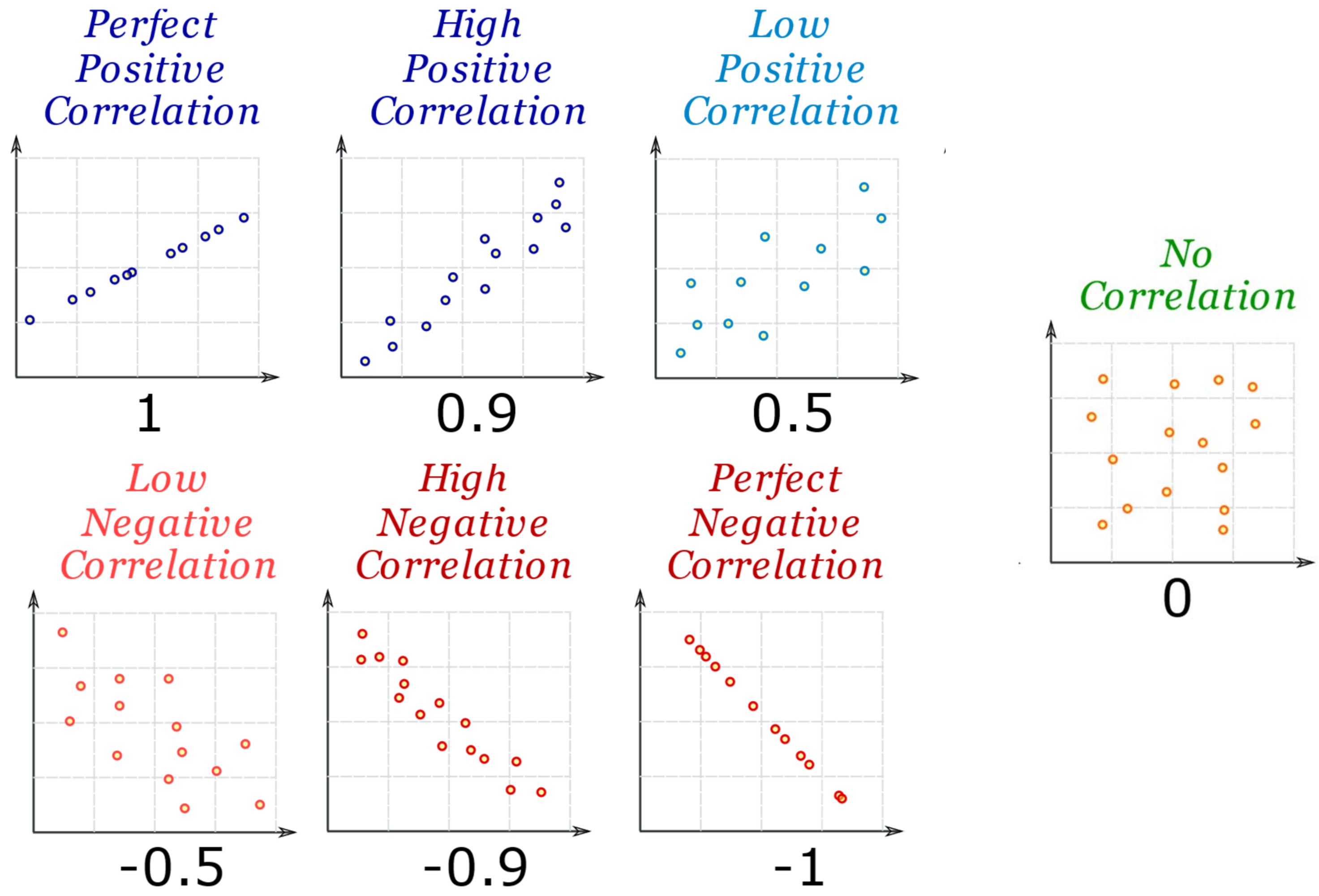
"Correlation"

Actually, this is "correlation coefficient"

- Correlation **measures a specific kind of dependency**.
  - Dependence = statistical relationship between two random variables (or two sets of data).
  - Correlation measures **"linear" relationship** between two random variables.



Correlation Coefficients

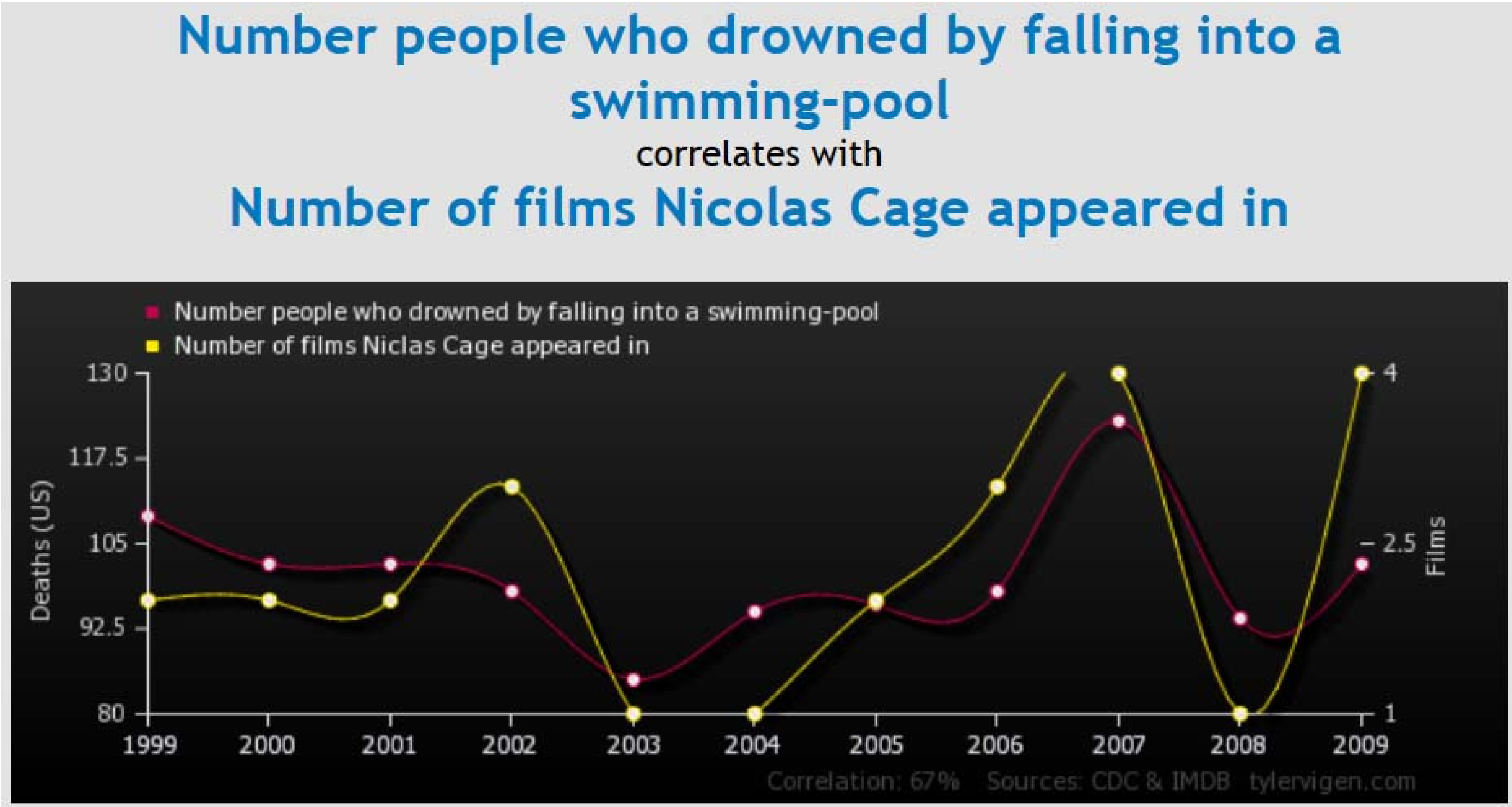


Correlation

Actually, this is "correlation coefficient"

- Correlation measures a specific kind of dependency.
  - Dependence = statistical relationship between two random variables (or two sets of data).
  - Correlation measures "linear" relationship between two random variables.
- Correlation and causality.
  - "Correlation does not imply causation"
  - Correlation cannot be used to infer a causal relationship between the variables.

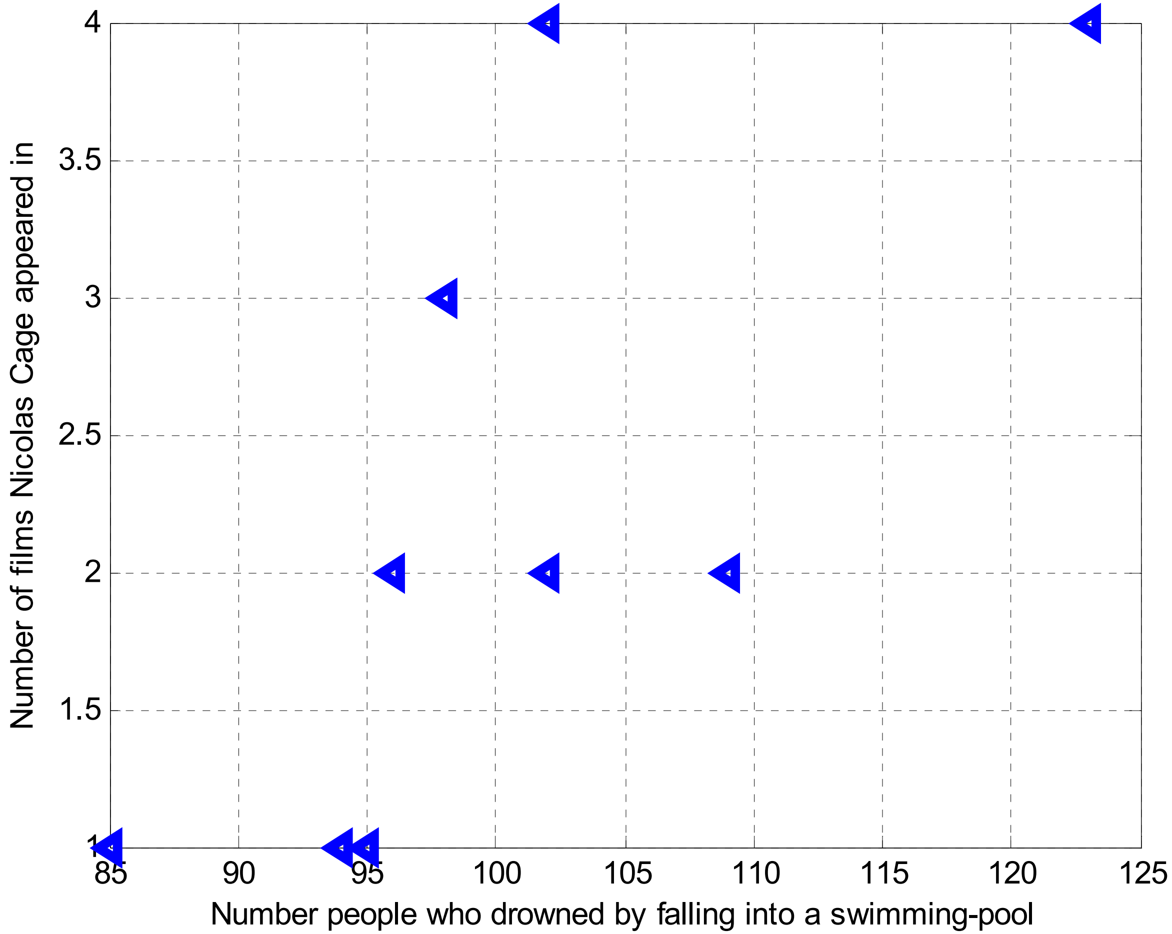
Two “Unrelated” Events



Correlation: 0.666004

http://www.tylervigen.com/

Two "Unrelated" Events



Correlation: 0.666004

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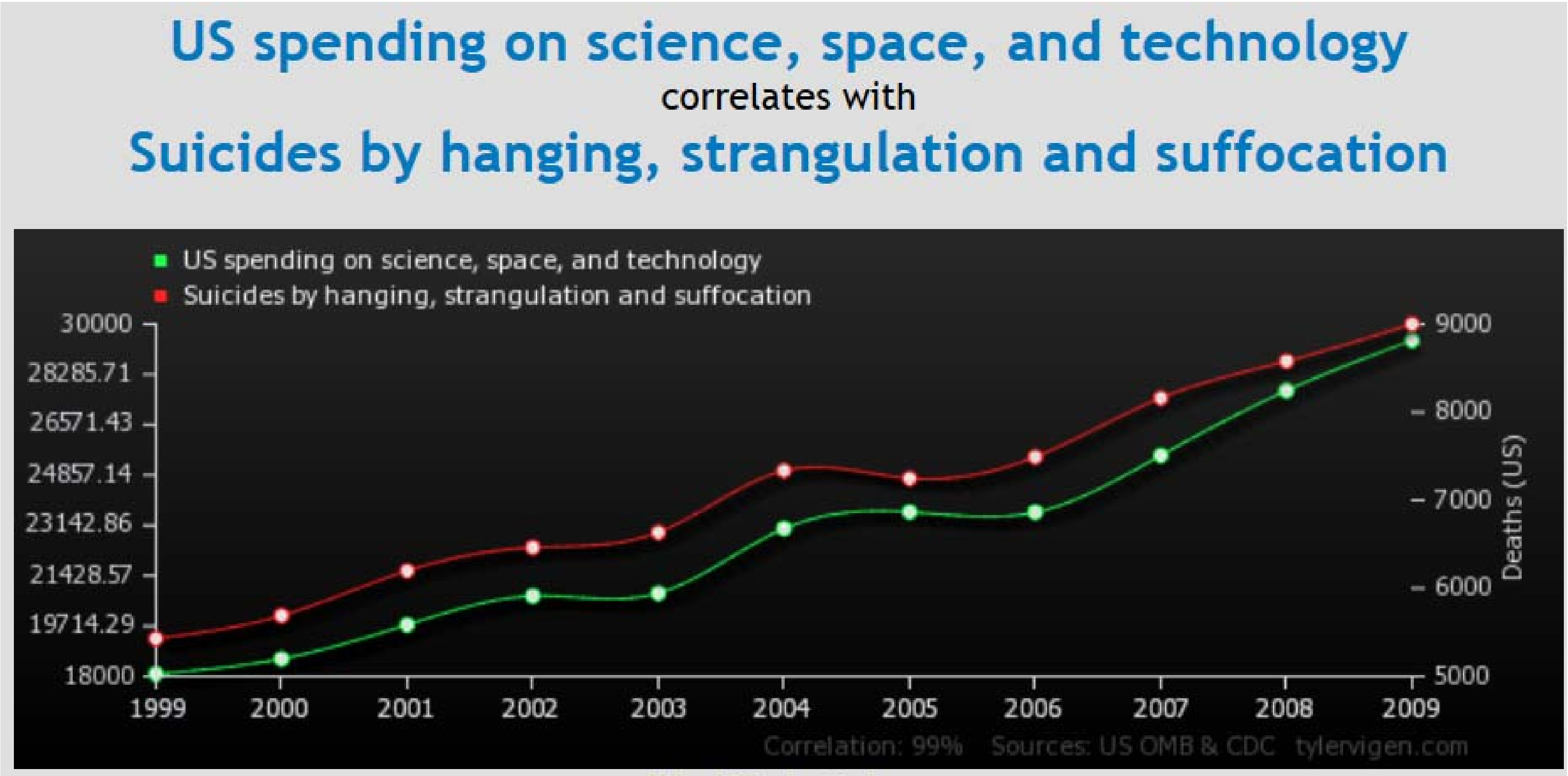
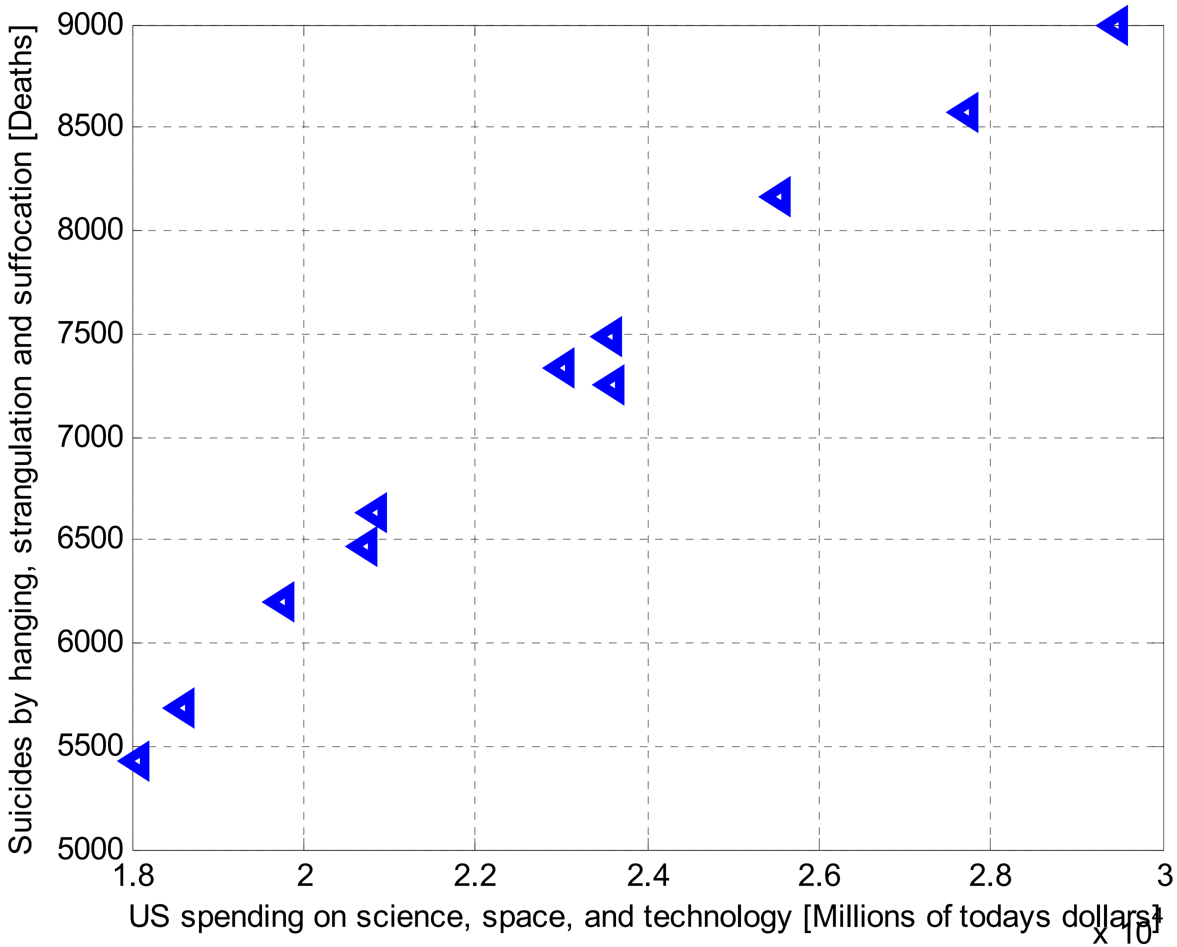
Spurious Correlation



Correlation: 0.992082

http://www.tylervigen.com/

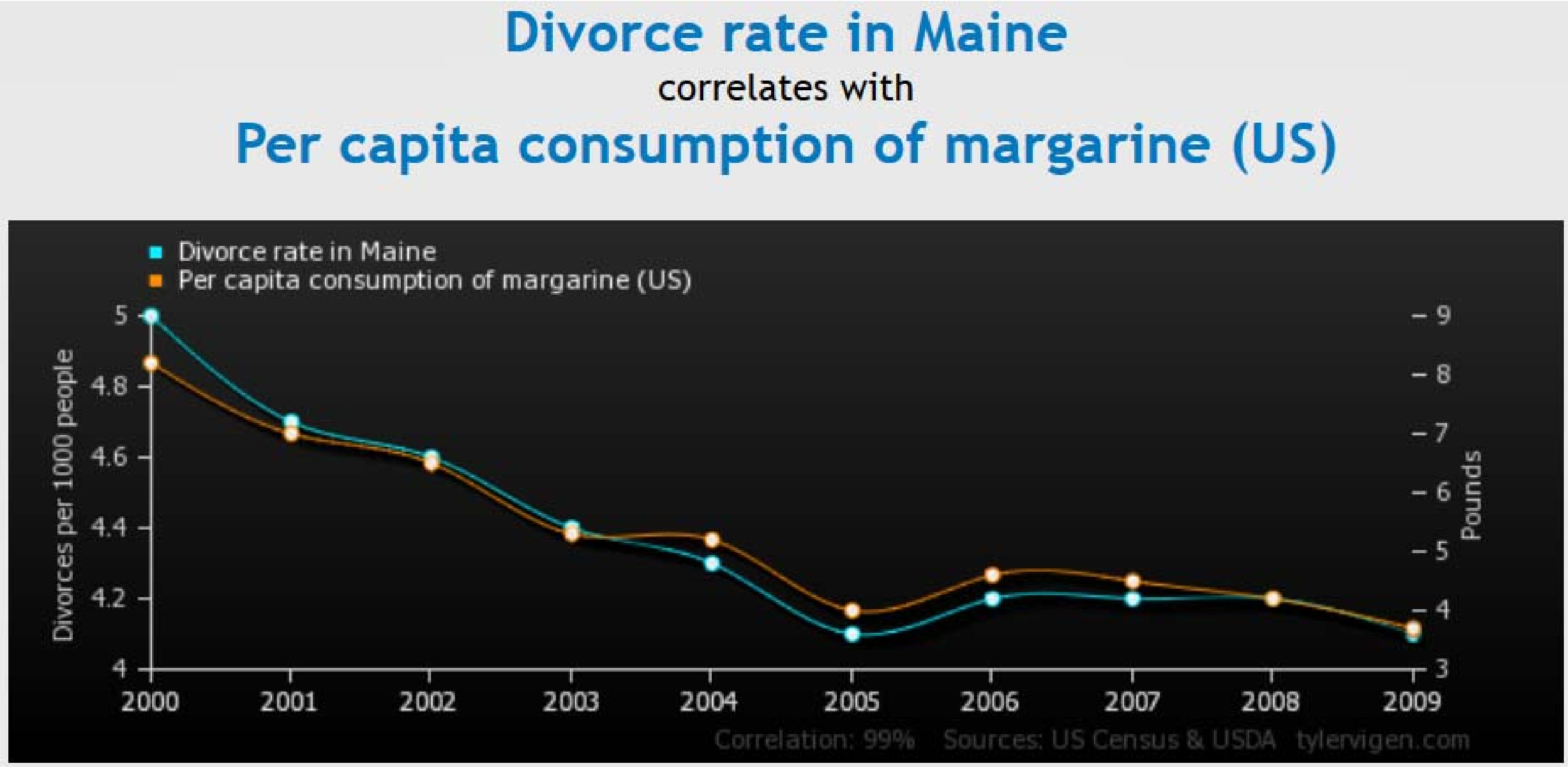
Spurious Correlation



Correlation: 0.992082

http://www.tylervigen.com/

Spurious Correlation



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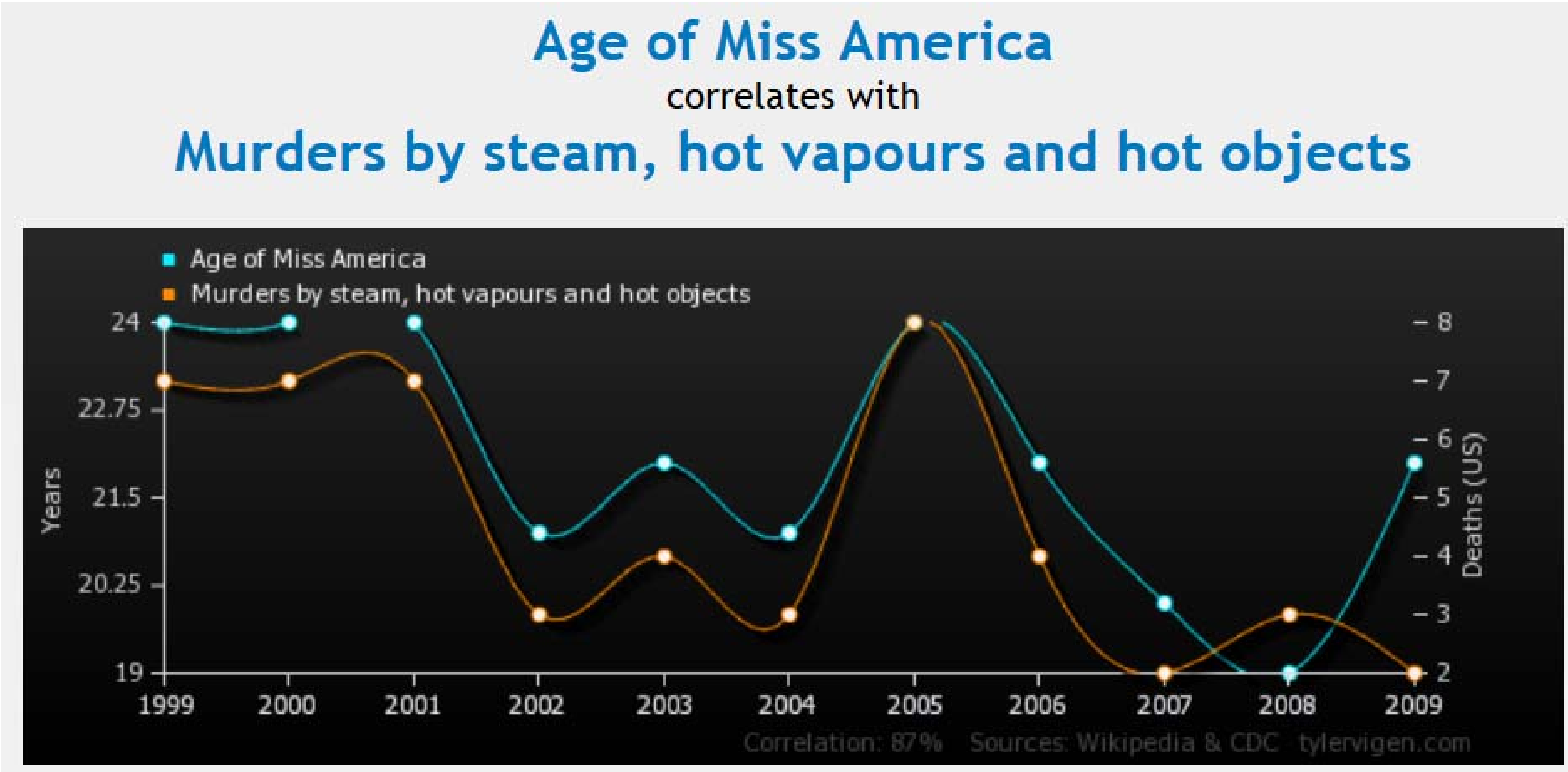
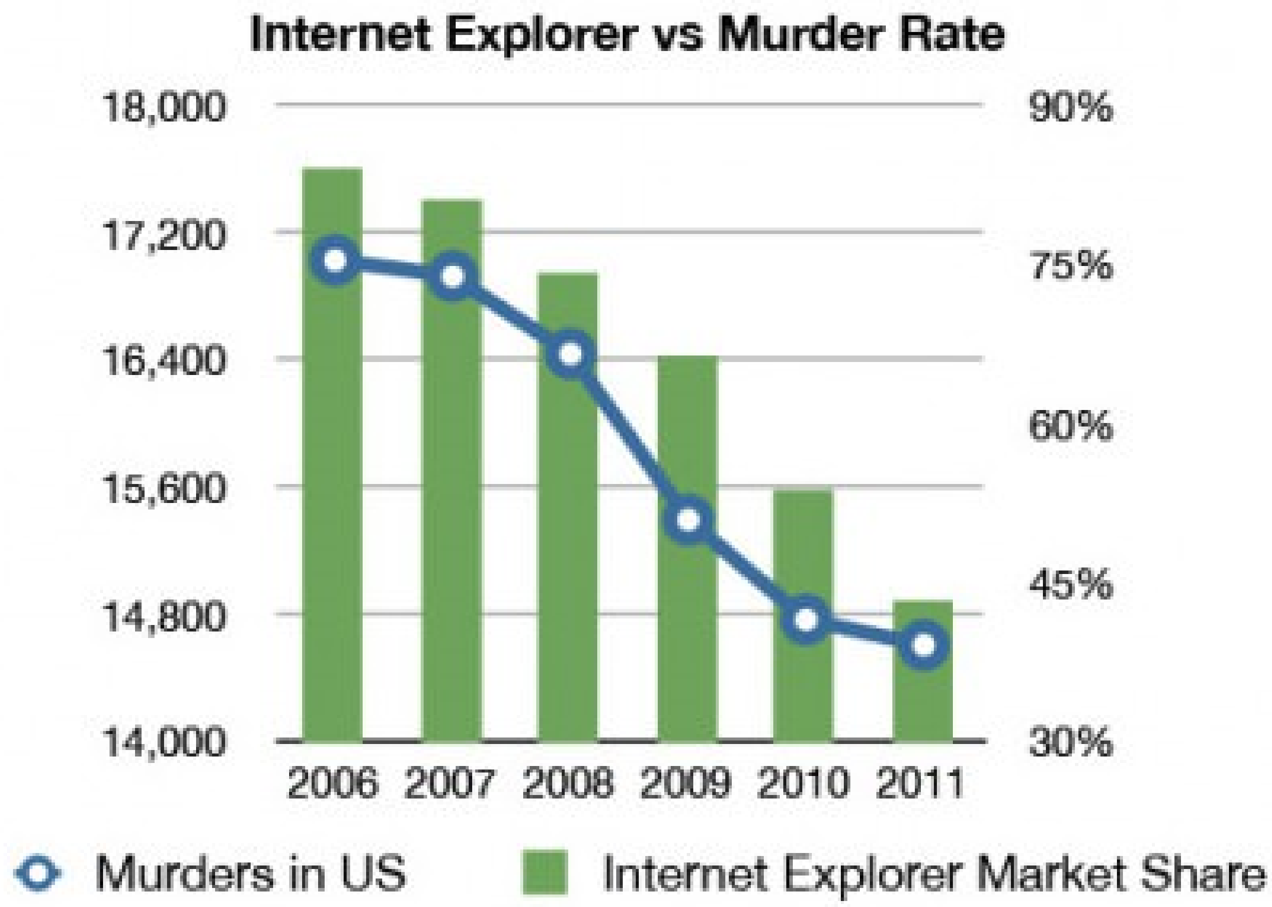
Spurious Correlation



http://www.tylervigen.com/

Spurious Correlation



(gross number of murders)

[http://www.geek.com/microsoft/does-internet-explorers-falling-market-share-mirror-the-drop-in-us-homicides-1537095/]

Spurious Correlation